THE CROSS SECTION MAP FOR GEODESIC FLOWS
RELATED TO THE HECKE AND PICARD GROUPS

by

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Abstract. — We consider the Hecke group $G_4$ and the Picard group $\text{PSL}(2, \mathbb{Z}[i])$. In the double of the quotient space of the upper half-space by $\text{PSL}(2, \mathbb{Z}[i])$, we find a double cover of the quotient surface of the upper half-plane by $G_4$. We analyze the cross section map of the geodesic flow on this surface by using the graphic method of Adler-Flatto.

Résumé (L’application de premier retour pour les flots géodésiques liés aux groupes de Hecke et de Picard)
Nous considérons le groupe de Hecke $G_4$ et le groupe de Picard $\text{PSL}(2, \mathbb{Z}[i])$. Dans le double de l’espace quotient du demi-espace supérieur par $\text{PSL}(2, \mathbb{Z}[i])$, on trouve un double revêtement de la surface quotient du demi-plan supérieur par $G_4$. Nous analysons l’application de premier retour du flot géodésique sur cette surface en utilisant la méthode graphique d’Adler-Flatto.

1. Introduction

The classical Markoff spectrum for $\mathbb{Q}$ has been studied from various points of view: continued fractions, quadratic forms and geometry. In [1], we gave a geometric interpretation of the Markoff spectrum for $\mathbb{Q}(i)$, generalizing the geometric study of the Markoff spectrum for $\mathbb{Q}$. A point we clarified is that the Picard group $\text{PSL}(2, \mathbb{Z}[i])$ naturally contains the Hecke group of order 4, $G_4$ (see §2), and that this subgroup captures the discrete part of the Markoff spectrum for $\mathbb{Q}(i)$.

In the present paper, we discuss the coding of the geodesic flow on a special surface in the double of the quotient space of the upper half-space $\mathbb{H}^3$ by $\text{PSL}(2, \mathbb{Z}[i])$. The special surface can be identified with a double cover of the quotient surface of the upper half-plane $\mathbb{H}^2$ by $G_4$. We show that the codings of the geodesic flows on the special surface and on $\mathbb{H}^2/G_4$ are characterized in a similar way (see §4). Thus, we find another ‘good’ feature of the Picard group depending on its subgroup $G_4$.

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To analyze a cross section map of the geodesic flow on the modular surface, R. Adler and L. Flatto used a graphic method in [2]. Let us summarize their method. The geodesic flow on the modular surface is coded by using the endpoints of the geodesic, which are denoted by \( \eta \) and \( \xi \). Since the universal cover of the modular surface is the upper half-plane \( \mathbb{H}^2 \) and its boundary is the real line, we have \( (\eta, \xi) \in (\mathbb{R} \cup \{\infty\})^2 \) and the geodesic is represented by the \( (\eta, \xi) \)-coordinate. Thus, the cross section map (first return map) can be simply expressed in the \( (\eta, \xi) \)-plane, where the cross section is the set of outward unit vectors whose base points are on the boundary of the usual fundamental domain of the modular group. Because of the shape of the domain, it is called the \emph{curvilinear map}. Even if this coding is natural, it does not have a Markovian partition. By a simple geometrical recoding, the \emph{rectilinear map} is obtained from the curvilinear map. Its domain is composed of rectilinear regions. The vertical and horizontal directions are contracting and expanding, respectively, under the rectilinear map. That is, the rectilinear map has a Markovian partition. Moreover, there is a conjugacy map between the curvilinear and the rectilinear maps which is the identity on most of its domain.

We apply Adler and Flatto’s method to analyze a cross section map of the geodesic flow on our surface, i.e., the immersion of the double cover of \( \mathbb{H}^2 / G_4 \) in the double of \( \mathbb{H}^3 / \text{PSL}(2, \mathbb{Z}[i]) \). Note that \( \mathbb{H}^2 / G_4 \) is one of the natural generalizations of the modular surface. We proceed as follows: in §3 we express the Poincaré return map of a cross section of the geodesic flow on our surface by using the endpoints of the geodesics (to define a curvilinear map \( T_C \)); in §4 we construct its linearized version (to define a rectilinear map \( T_R \)), and find a conjugacy map \( \Phi \) between them. The conjugacy map \( \Phi \) is the identity on most of the set on which it is defined. (The same situation arises with the geodesic flow on the modular surface.) This fact is stated in Theorem 4.1, which is our main result. Note that the cross section map we obtain corresponds to a continued fraction expansion of complex numbers whose partial quotients are of the form \( k(1 + i) \) and \( k(1 - i) \), \( k \in \mathbb{Z} \).

In the construction of the conjugacy map, we clarify the relations between some of the generators of the Picard group. The graphs (see Figures 2 and 3) used in this paper are more complicated than the corresponding ones for the modular surface (see [2]). That means that a Markovian partition of our geodesic flow is more complicated than the one of the geodesic flow in the modular surface. An interest of the graphic method is that it makes the difference intelligible.

For the geodesic flow in the quotient surface \( \mathbb{H}^2 / G_4 \), it is also possible to construct the curvilinear map, the rectilinear map and the conjugacy map between them. The construction of these maps is almost the same, as we produce in §3 and §4. We only write down these maps in a remark after Theorem 4.1. Note that these maps and their graphs are almost the same as the ones for the geodesic flow on the special surface in the 3-manifold.

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2. Hecke group, Picard group

We begin by introducing the groups and spaces which we will use in this paper. We always identify a point \((x,y)\) in \(\mathbb{R}^2\) with \(x + y i\) and a point \((x,y,t)\) in \(\mathbb{R}^3\) with \(x + y i + t j\), where \(i^2 = j^2 = -1\). The upper half-plane \(\{ z = x + iy \in \mathbb{C} \mid y > 0 \}\) is denoted by \(\mathbb{H}^2\) and the upper half-space \(\{ z +jt \mid z \in \mathbb{C}, t > 0 \}\) is denoted by \(\mathbb{H}^3\). Suppose that they are endowed with the hyperbolic metrics \(ds^2 = (dx^2 + dy^2)/y^2\) and \(ds^2 = (dx^2 + dy^2 + dt^2)/t^2\), respectively.

The Hecke group \(G_4\) is generated by two elements (see [5])

\[
G_4 := \left\langle A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad P := \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix} \right\rangle.
\]

The Picard group \(\Gamma\) is generated by \(A, T, U\) and \(L\). It is denoted by \(\Gamma = \langle A, T, U, L \rangle\), where

\[
T := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U := \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \quad L := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]

The Picard group is also represented by \(\Gamma = \text{PSL}(2, \mathbb{Z}[i])\), where \(\mathbb{Z}[i]\) is the set of complex integers (see [4]). The group \(G_4\) acts on \(\mathbb{H}^2\) by fractional linear transformations. The group \(\Gamma\) acts on \(\mathbb{C}\) by fractional linear transformations and on \(\mathbb{H}^3\) by their Poincaré extensions (see [3]). In what follows we always identify an element \(g \in G_4\) with the fractional linear transformation and an element \(g \in \Gamma\) with the fractional linear transformation or its Poincaré extension induced by \(g\).

The Hecke group \(G_4\) acts on \(\mathbb{H}^2\) discontinuously and a fundamental domain of \(G_4\) can be represented as follows:

\[
F_4 := \left\{ x + iy \in \mathbb{H}^2 \left| x^2 + y^2 > 1, \quad |x| < \frac{1}{\sqrt{2}} \right. \right\},
\]

where \(|x|\) denotes the absolute value of \(x\). Topologically this is the same as the modular surface, that is, a sphere minus a point. There are two singular points on the quotient surface \(F_4\): \(i\) and \((-1+i)/\sqrt{2}\), this latter is identified with \((1+i)/\sqrt{2}\). Their ramification numbers are 2 and 4, respectively, which come from \(A^2 = (AP)^4 = \text{Id}\).

The latter is different from the singular point on the modular surface coming from \((AT)^4 = \text{Id}\).

The Picard group \(\Gamma\) acts on \(\mathbb{H}^3\) discontinuously and a fundamental region of \(\Gamma\) can be represented as follows:

\[
F := \left\{ x + iy + jt \in \mathbb{H}^3 \left| x^2 + y^2 + t^2 > 1, \quad |x| < \frac{1}{2}, \quad 0 < y < \frac{1}{2} \right. \right\}.
\]

The region \(F\) has a single parabolic vertex at \(\infty\) and has a finite volume. (See Example 15 in [6].) Then \(\tilde{F} = F \cup LF\) is a fundamental region of \(\langle A, T, U \rangle\). We call \(\tilde{F}\) the \textit{fundamental polyhedron}.

Define lines \(l_1, l_2\) on \(\mathbb{C}\) as \(l_1 := \{(x, y, 0) \mid x = y\}, \quad l_2 := \{(x, y, 0) \mid x = -y\}\) and define planes \(\tilde{W}_1, \tilde{W}_2\) in \(\mathbb{H}^3\) as \(\tilde{W}_1 := \{(x, y, t) \mid x = y, t > 0\}, \quad \tilde{W}_2 := \{(x, y, t) \mid x = -y, t > 0\}\). Note that \(l_1, l_2\) are boundaries of \(\tilde{W}_1, \tilde{W}_2\), respectively. Take the following
two matrices:

$$M_1 := \begin{pmatrix} \frac{1}{\sqrt{2}} (1+i) & 0 \\ 0 & 1 \end{pmatrix}, \quad M_2 := \begin{pmatrix} \frac{1}{\sqrt{2}} (1-i) & 0 \\ 0 & 1 \end{pmatrix}.$$ 

They give identifications between the planes $\hat{W}_1$, $\hat{W}_2$ and the upper half-plane $\mathbb{H}^2$. Indeed,

$$\mathbb{H}^2 \ni x + it \leftrightarrow M_1(x + jt) = \frac{1}{\sqrt{2}}(1+i)x + jt \in \hat{W}_1,$$

$$\mathbb{H}^2 \ni x + it \leftrightarrow M_2(x + jt) = \frac{1}{\sqrt{2}}(1-i)x + jt \in \hat{W}_2.$$ 

Under this identification, if we use the coordinates $x + it$, $\hat{W}_1$ and $\hat{W}_2$ are denoted simply by $W_1$ and $W_2$, respectively. Define $k = k+1 \pmod 2$, $k \in \{1, 2\}$ for $k \in \{1, 2\}$.

We consider the action on $\hat{W}_k$ defined by the following matrices:

$$P_1^\pm = \pm TU = \begin{pmatrix} 1 & \pm(1+i) \\ 0 & 1 \end{pmatrix}, \quad P_2^\pm = \pm TU^{-1} = \begin{pmatrix} 1 & \pm(1-i) \\ 0 & 1 \end{pmatrix}.$$ 

**Lemma 2.1.** — (i) The action of $A$ switches the planes $\hat{W}_1$ and $\hat{W}_2$, that is, $AW_k = \hat{W}^g_k$.

(ii) The action $P_k^\pm$ is a parallel displacement on $W_k$ and satisfies $P_k^\pm \hat{W}_k = \hat{W}_k$.

(iii) The action $P_k^\pm$ on $\hat{W}_k$ is equivalent to the action $P$ on $W_k$.

**Proof.** — The assertions (i) and (ii) are easily checked by a calculation. For (iii) we can easily check that for $x + it \in \mathbb{H}^2$, $P(x + it)$ is identified with $P_k^\pm M_k(x + jt)$. 

The intersection of $\hat{W}_1 \cup \hat{W}_2$ with the fundamental polyhedron $\hat{F}$ can be identified with two sheets of the fundamental domain $F_4$, that is, $F_{14} \cup F_{12}$, where $F_{ik}$ denotes the domain $F_k$ on $W_k$. Moreover, $F_{41} \cap F_{12}$ and the boundary of $F_{41} \cup F_{12}$ are the lines from the singular points to $\infty$, i.e., the vertical lines which lie over $0 \in \mathbb{C}$ and $(1+i)/2$. The point $(1+i)/2$ is identified with $(-1+i)/2$, $(1-i)/2$ and $(1-i)/2$ by the action of $T$ and $U$.

### 3. Cross section

The geodesics in $\mathbb{H}^2$ are the semicircles and the straight lines orthogonal to the real axis. Let $T_1 \mathbb{H}^2$ denote the unit tangent bundle of $\mathbb{H}^2$. An element $u$ in $T_1 \mathbb{H}^2$ has coordinates $(x, y, \theta)$, where $(x, y)$ are coordinates of the base point of $u$ and $\theta$ is the angle between $u$ and a horizontal line. For $u \in T_1 \mathbb{H}^2$ there exists a unique geodesic tangent to $u$ and passing through the base point of $u$. We always define a direction of the geodesic tangent to $u$ as the direction of $u$. There exists another coordinate system for $w$: $(\eta, \xi, s)$, where $\eta$ and $\xi$ are the negative and positive endpoints of the geodesic, respectively, and $s$ is the arc length from the center of the geodesic to the base point of $u$. The hyperbolic measure on $T_1 \mathbb{H}^2$ is represented by

$$dm = \frac{dxdy \theta}{y^2} = \frac{2d\xi ds}{(\xi - \eta)^2}.
We define the geodesic flow $g_t$ by mapping an element $u \in T_1H^2$ to the unit vector $g_t(u)$ tangent to the geodesic determined by $u$ with base point at a hyperbolic length $t$ ahead. By using the second coordinate system, the geodesic flow is described as $g_t(\eta, \xi, s) = (\eta, \xi, s + t)$.

Since $\hat{F} \cap (W_1 \cup W_2) = F_{41} \cup F_{42}$, the geodesic flow $g_t$ on $T_1H^2$ induces a flow on $T_1F_{41} \cup T_1F_{42}$, where $T_1F_{4k}$ is the unit tangent bundle of $F_{4k}$. We describe the boundary of $F_{4k}$ as follows:

$$S_{k1} = \{(x, y) \in H^2 \mid x = \frac{1}{\sqrt{2}}, y > \frac{1}{\sqrt{2}}\}, \quad S_{k2} = \{(-x, y) \in H^2 \mid (x, y) \in S_{k1}\},$$

$$S_{k3} = \{(x, y) \in H^2 \mid x^2 + y^2 = 1, -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}\}.$$

A vector on $F_{4k}$ flows along the geodesic to which it is tangent until its base point hits the boundary of $F_{4k}$. There are two cases of identifications of vectors on boundaries. The first case: since $S_{k1}$ and $S_{k2}$ are identified by $P$, the exiting vectors on $S_{k1}$ and $S_{k2}$ are equivalent to entering vectors on $S_{k2}$ and $S_{k1}$, respectively. These vectors determine new geodesics in $F_{4k}$ along which they continue moving. The second case: since $S_{k3}$ and $S_{k4}$ are identified by $A$ and the action of $A$ switches $W_1$ and $W_2$, an exiting vector on $S_{k3}$ is equivalent to an entering vector on $S_{k4}$ and it also determines a new geodesic in $F_{4k}$ along which it continues moving. Note that in the first case the geodesic flow always exists in $F_{4k}$, but in the second case the geodesic flow in $F_{4k}$ move to the one in $F_{4k}$.

Let $S_{k1}^{\text{into}}$ and $S_{k4}^{\text{out}}$ be the set of geodesics cutting $S_{k1}$, going into and out of $F_{4k}$, respectively. The sets of geodesics $S_{k1}^{\text{into}}, S_{k2}^{\text{out}}, l = 2, 3$ are defined in the same way.

Now we can naturally define a cross section as the set $C_k$ of outward-directed unit tangent vectors with base points on the boundary of $F_{4k}$. The set $C_k$ is naturally partitioned as $C_k = C_{k1} \cup C_{k2} \cup C_{k3} \cup C_{k4}$ where

$$C_{k1} = S_{k1}^{\text{out}}, \quad C_{k2} = S_{k1}^{\text{into}} \cap S_{k1}^{\text{out}}, \quad C_{k3} = S_{k1}^{\text{into}} \cap S_{k4}^{\text{out}}, \quad C_{k4} = S_{k4}^{\text{out}}.$$

We introduce the extended line $L = [-\infty, -0] \cup [0, +\infty]$ with $-0, -\infty$ distinct from $+0, +\infty$, respectively. We define the extended plane as the Cartesian product $L \times L$. The reason for introducing signed zeros and infinities will be clarified later. Let $\eta$ and $\xi$ denote the negative and positive endpoints of a geodesic in $H^2$, respectively. If the geodesic is in $W_{4k}$, these points are denoted by $(\eta, \xi)_{k}$. Considering the boundary of $W_{4k}$ as the extended line $L$, we get $(\eta, \xi)_k \in L \times L$. Denote by $E_k$ the extended plane where $(\eta, \xi)_k$ exists. The partition of the cross section in $E_k$ can be described as follows (see Figure 1):

$$C_{k1} = \{(\eta, \xi) \mid (\eta, \xi) \in \left(\eta - \frac{1}{\sqrt{2}}, \eta + \frac{1}{\sqrt{2}}\right) \cap \left(\xi - \frac{1}{\sqrt{2}}, \xi + \frac{1}{\sqrt{2}}\right) \text{ and } \eta < \xi\}$$

$$C_{k2} = \{(\eta, \xi) \mid (\eta, \xi) \in \left(\eta - \frac{1}{\sqrt{2}}, \eta + \frac{1}{\sqrt{2}}\right) \cap \left(\xi - \frac{1}{\sqrt{2}}, \xi + \frac{1}{\sqrt{2}}\right) \text{ and } \eta < \xi\}$$

$$C_{k3} = \{(-\eta, -\xi) \mid (\eta, \xi) \in C_{k2}\}, \quad C_{k4} = \{(-\eta, -\xi) \mid (\eta, \xi) \in C_{k1}\}.$$

Note that the endpoints of geodesics passing through $(1+i)/\sqrt{2}$ which is the boundary point of $S_{k1}$ and $S_{k3}$ and through $(-1+i)/\sqrt{2}$ which is the boundary point of $S_{k2}$.
with respect to the boundary of respectively. This is why these hyperbolas give the boundary of the partition. The map \( E \) in

To describe precisely the images, we introduce the following partitions: \( S \) and \( S']\) are not on the same extended planes where \( k \) means the image by \( T \). Moreover, the domain \( S \) and \( S' \) are adjacent in \( S \). A prime \( t \) means the image by \( T \), for example, \( C'_1 \) is the image of \( C_1 \) by \( T \). Note that the map \( T \) on \( C_{k2}^+\) switches the extended planes, so in the range their images are denoted by \( C_{k2}^{+/0}\). Figure 1

and \( S_{k3} \) are the graphs of the equations

respectively. This is why these hyperbolas give the boundary of the partition.

Then we obtain the following cross section map, that is, the Poincaré return map with respect to the boundary of \( F_{2k} \), that is, \( S_{k1} \cup S_{k2} \cup S_{k3} \):

From the shape of the domains, we call \( C_k \) the \textit{curvilinear region} and this map the \textit{curvilinear map}. The graph of this map (Figure 1) is almost the same as the one of the cross section map for the geodesic flow on the modular surface (see §4 in [2]). The difference between this and ours is that the images of \( C_{12}, C_{13}, C_{22} \) and \( C_{23} \) by \( T \) are not on the same extended planes where \( C_{12}, C_{13}, C_{22} \) and \( C_{23} \) exist, respectively. To describe precisely the images, we introduce the following partitions:

The map \( T \) on \( C_{k2}^+\), \( C_{k3}^+ \) switches the extended planes, that is, the images of them are in \( E_k \). Moreover, the domain \( C_{k2}^+ \) and \( C_{k3}^+ \) are adjacent in \( E_k \), respectively, but their
The left and right sides are the domain and the range of \( T_R \), respectively. A prime denotes the image by \( T_R \), for example, \( R'_{k1} \) is the image of \( R_{k1} \) by \( T_R \).

Note that the map \( T_R \) on \( R_{k2}, R_{k3} \) switches the extended planes, so in the range their images are denoted by \( R'_{k2}, R'_{k3} \).

Figure 2

images are not adjacent in \( E_k \) (see Figure 1). This is the reason we have introduced the distinct \( \pm 0 \).

Let us introduce the following coordinates:

\[
(\eta, \xi)_1 := \left( \frac{1}{\sqrt{2}}(1 + i)\eta, \frac{1}{\sqrt{2}}(1 + i)\xi \right), \quad (\eta, \xi)_2 := \left( \frac{1}{\sqrt{2}}(1 - i)\eta, \frac{1}{\sqrt{2}}(1 - i)\xi \right).
\]

Then we can consider the curvilinear map as a map on \((L_1 \times L_1) \cup (L_2 \times L_2)\) where \( L_1 \) and \( L_2 \) are the extended lines corresponding to \( l_1 \) and \( l_2 \), respectively. It is described as:

\[
\tilde{T}_C(\eta, \xi)_k = \begin{cases} 
P_k^-(\eta, \xi)_k & \text{if } (\eta, \xi)_k \in C_{k1}, \\
A(\eta, \xi)_k & \text{if } (\eta, \xi)_k \in C_{k2} \cup C_{k3}, \\
P_k^+(\eta, \xi)_k & \text{if } (\eta, \xi)_k \in C_{k4}, 
\end{cases}
\]

where the action of a matrix \( M \) is \( M(\hat{\eta}, \hat{\xi}) = (M\hat{\eta}, M\hat{\xi}) \) for \( \hat{\eta}, \hat{\xi} \in \mathbb{C} \).

4. Conjugacy

The aim of this section is to construct a map \( T_R \) conjugate to the curvilinear map \( T_C \). The map \( T_R \) is defined on a region obtained by straightening out the curvilinear region \( C_k \). The point is that the partition of \( C_k \) is not Markovian, but for the domain of \( T_R \) we can have a Markovian partition. Moreover, we can define \( T_R \) such that \( T_R = T_C \) on most of the set where they are defined. We will largely follow the argument of §6 in [2].

\[ \]
For the domain of \( T_R \), let us introduce a set \( R_k \) with partition in the extended plane \( E_k \): \( R_k = \bigcup_{l=1}^{6} R_{kl} \), where we define \( R_{kl} \), \( l = 1, \ldots, 6 \) by straightening out the set \( C_{kl} \), \( l = 1, \ldots, 4 \) (see the left graphs in Figures 1, 2),

\[
R_{k1} = \left\{ (\eta, \xi) | \eta \leq -1, \ 1 < \xi \leq \sqrt{2} \right\} \\
\quad \cup \left\{ (\eta, \xi) | -\infty < \eta \leq -\sqrt{2} - 1, \ \sqrt{2} < \xi \leq +\infty \right\},
\]

\[
R_{k2} = \left\{ (\eta, \xi) | -\infty \leq \eta \leq -1 - \sqrt{2}, \ 1 - \sqrt{2} < \xi \leq -0 \right\},
\]

\[
R_{k2}^+ = \left\{ (\eta, \xi) | -\infty \leq \eta \leq -1, \ +0 \leq \xi < 1 \right\}, \ R_{k2} = R_{k2}^+ \cup R_{k2}^+,
\]

\[
R_{k3}^+ = \left\{ (-\eta, -\xi) | (\eta, \xi) \in R_{k2}^+ \right\}, \ R_{k3} = R_{k3}^+ \cup R_{k3}^+,
\]

\[
R_{k4} = \left\{ (-\eta, -\xi) | (\eta, \xi) \in R_{k1} \right\},
\]

\[
R_{k5} = \left\{ (\eta, \xi) | -1 < \eta \leq -0, \ 1 < \xi < \sqrt{2} \right\}, \ R_{k6} = \left\{ (-\eta, -\xi) | (\eta, \xi) \in R_{k5} \right\}.
\]

Then we define the map \( T_R \) of \( R_k \) on itself as

\[
T_R(\eta, \xi) = \begin{cases} 
(\eta - \sqrt{2}, \xi - \sqrt{2})_k & \text{on } R_{k1}, \\
\left( \frac{-1}{\eta}, \frac{1}{\xi} \right)_k & \text{on } R_{k2}, R_{k3}, \\
(\eta + \sqrt{2}, \xi + \sqrt{2})_k & \text{on } R_{k4}, \\
\left( \frac{-\sqrt{2}\eta + 1}{\eta}, \frac{-\sqrt{2}\xi + 1}{\xi} \right)_k & \text{on } R_{k5}, \\
\left( \frac{\eta}{\sqrt{2}\eta + 1}, \frac{\xi}{\sqrt{2}\xi + 1} \right)_k & \text{on } R_{k6}.
\end{cases}
\]

In the same way as before, from the shape of the domain, we call \( R_k \) the rectilinear region and \( T_R \) the rectilinear map.

If the domain of a first return map can be partitioned into rectangles which are the product of pieces of expanding and contracting directions, and such that the first return map maps each rectangle exactly across several rectangles in the expanding direction, we say that the first return map has the Markov property and its partition is Markovian.

**Proposition 4.1.** — The map \( T_R \) has the Markov property.

*Proof.* — Vertical (resp. horizontal) lines in \( R_{k2}, R_{k3}, R_{k5} \) and \( R_{k6} \) are mapped by \( T_R \) to vertical (resp. horizontal) lines in their images. (See Figure 2.) Clearly, the vertical direction is contracting and the horizontal direction is expanding. Let us consider \( R_{k2} \). It consists of the two rectangles \( R_{k2}^-, R_{k2}^+ \) which are the product of lines of expanding and contracting directions. The image \( T_R(R_{k2}^-) \) is in \( R_{k1} \). The image \( T_R(R_{k2}^+) \) crosses \( R_{k4} \) and \( R_{k6} \) in the expanding direction. In the same way, we can verify that \( R_{k3}, R_{k5} \) and \( R_{k6} \) are Markovian.

On \( R_{k1} \) and \( R_{k4} \), the map \( T_R \) is a parallel displacement. Let us consider the vertical direction as the contracting one and the horizontal direction as the expanding one. The region \( R_{k1} \) can be regarded as the two rectangles and \( T_R(R_{k1}) \) crosses \( R_{k2}, R_{k2}^+ \) and \( R_{k1} \) in the expanding direction. The discussion is the same for \( R_{k4} \). \( \square \)
The rectilinear map can be also described in the following manner by using the action of matrices:

\[
\tilde{T}_R(\eta, \xi)_k = \begin{cases} 
    P_k(\eta, \xi)_k & \text{if } (\eta, \xi)_k \in R_{k1}, \\
    A(\eta, \xi)_k & \text{if } (\eta, \xi)_k \in R_{k2} \cup R_{k3}, \\
    P_k^+ (\eta, \xi)_k & \text{if } (\eta, \xi)_k \in R_{k4}, \\
    P_k^+ AP_k^+ (\eta, \xi)_k & \text{if } (\eta, \xi)_k \in R_{k5}, \\
    P_k^+ AP_k^+ AP_k^+ (\eta, \xi)_k & \text{if } (\eta, \xi)_k \in R_{k6}.
    \end{cases}
\]

The matrices occurred in the map above are explicitly:

\[
P_1^+ AP_2^+ AP_1^+ = \begin{pmatrix} 1 & 0 \\
\pm(1-i) & 1 \end{pmatrix}, \quad P_2^+ AP_1^+ AP_2^+ = \begin{pmatrix} 1 & 0 \\
\pm(1+i) & 1 \end{pmatrix}.
\]

Next we define a conjugacy map between \(T_C\) and \(T_R\). Note that the curvilinear region \(C_k\) and its partition, and the rectilinear region \(R_k\) and its partition are similar but not identical and that most of \(C_k\) overlaps with most of \(R_k\) (see Figure 3). We will construct a one-to-one map \(\Phi\) from \(C_1 \cup C_2\) onto \(R_1 \cup R_2\) such that \(T_R \circ \Phi = \Phi \circ T_C\) and \(\Phi = \text{id}\) on \((C_1 \cap R_1) \cup (C_2 \cap R_2)\), that is, \(\Phi\) is the identity on most of the set where it is defined.

Let us introduce some notations in order to describe the difference between \(C_k\) and \(R_k\) (see Figure 3): \(C_k - R_k = U_{k1}^+ \cup V_{k1}^+ \cup V_{k2}^- \cup U_{k1}^- \cup V_{k1}^-\), where

\[
U_{k1}^+ = \left\{ (\eta, \xi)_k \in C_{k3} - R_k \cap C_k \mid \eta > 1 + \sqrt{2} \right\},
\]

\[
V_{k1}^+ = \left\{ (\eta, \xi)_k \in C_{k3} - R_k \cap C_k \mid \eta < 1 + \sqrt{2} \right\},
\]

\[
U_{k2}^+ = \left\{ (\eta, \xi)_k \in C_{k2} - R_k \cap C_k \mid \eta < -1 + \sqrt{2} \right\},
\]

\[
V_{k1}^- = \left\{ (\eta, \xi)_k \in C_{k2} - R_k \cap C_k \mid \eta > -1 + \sqrt{2} \right\},
\]

\[
V_{k2}^- = C_{k1} - R_k \cap C_k,
\]

and \(R_k - C_k = V_{k1}^- \cup V_{k2}^- \cup U_{k1}^- \cup V_{k1}^- \cup V_{k2}^- \cup U_{k1}^-\) where

\[
\tilde{V}_{k1}^+ = R_{k4} - R_k \cap C_k, \quad \tilde{V}_{k2}^+ = R_{k3} - R_k \cap C_k, \quad \tilde{U}_{k1}^- = R_{k6} - R_k \cap C_k,
\]

\[
\tilde{V}_{k1}^- = R_{k1} - R_k \cap C_k, \quad \tilde{V}_{k2}^- = R_{k2} - R_k \cap C_k, \quad \tilde{U}_{k1}^+ = R_{k5} - R_k \cap C_k.
\]

We will arrange that a set \(\tilde{W}\) is the image of a set \(W\) by \(\Phi\), where \(W = U_{k1}^+, V_{k1}^+, V_{k2}^-\).

We consider the images of \(U_{k1}^+\) in the following diagram.
The dotted line segments represent the curvilinear region $C_k$ and the line segments represent the rectilinear region $R_k$ in the extended plane $E_k$. We can observe that their intersection is most of $C_k$ and of $R_k$.

Figure 3

In the diagram the first line represents transformations by $T_C$ and the second line transformations by $T_R$. We want to construct a transformation $\Phi$ from the first line to the second. Since $T_C^{-1}U_{11}^+$ is in $C_{14} \cap R_{13}$, we define $U_{10}^+ = U_{10}^+$ as $T_C^{-1}U_{11}^+$, where $T_C = P_{1,1}^+$, that is, $\Phi$ is the identity on $U_{10}^+$. Since $T_CU_{11}^+ \subset C_{24} \cap R_{24}$, we define $U_{12}^+ = U_{12}^+$ as $T_CU_{11}^+$, where $T_C = A$, that is, $\Phi$ is also the identity on $U_{12}^+$. We have already defined $\overline{U_{13}^+}$ such that $T_R\overline{U_{10}^+} = \overline{U_{11}^+}$ where $T_R = A$. Moreover, it can be checked that $T_RU_{11}^+ = U_{12}^+$ where $T_R = P_2^-AP_1^-AP_2^-$. Then the map $\Phi$ from $U_{11}^+$ to $\overline{U_{11}^+}$ can be defined so that the diagram commutes. Thus, we obtain $\Phi = AP_1^-$ on
Indeed, we can directly check \( A = (P_2^- AP_1^- AP_2^-)(AP_1^+) \) which is equivalent to \( (P_2^- AP_1^- A)^2 = Id \).

In the same way, we can define the map \( \Phi \) on \( U_{11}^+, U_{21}^+ \) and \( U_{21}^- \), using \( A = (P_2^+ AP_1^+ AP_2^+)(AP_1^+) \), \( A = (P_2^- AP_2^- AP_1^-)(AP_2^-) \) and \( A = (P_1^+ AP_2^- AP_1^-)(AP_2^+) \), respectively. They are \( \Phi = AP_k^+ \) on \( U_{11}^+ \), where we use the same notation as the maps \( T_C, T_R \) and the matrices are

\[
AP_1^+ = \begin{pmatrix} 0 & -1 \\ 1 & \pm(1+i) \end{pmatrix}, \quad AP_2^+ = \begin{pmatrix} 0 & -1 \\ 1 & \pm(1-i) \end{pmatrix}.
\]

Next we consider the images of \( V_{11}^+ \) in the following diagram.

\[\begin{array}{cccccc}
V_{10}^+ & \xrightarrow{P_1^+} & V_{11}^+ & \xrightarrow{A} & V_{12}^+ & \xrightarrow{P_2^+} & V_{13}^+\\
\downarrow{id} & & \downarrow{P_1^+ AP_2^+ A} & & \downarrow{AP_2^+} & & \downarrow{id} \\
V_{10}^- & \xrightarrow{P_1^+ AP_2^+ AP_1^+} & V_{11}^- & \xrightarrow{A} & V_{12}^- & \xrightarrow{P_1^-} & V_{13}^-
\end{array}\]

In the same way as before, the first line represents transformations by \( T_C \) and the second line transformations by \( T_R \). Since \( T_C^{-1} V_{11}^+ \) is in \( C_{14} \cap R_{16} \), we define \( V_{10}^+ = \overline{V_{10}^+} \) as \( T_C^{-1} V_{11}^+ \), where \( T_C = P_1^+ \). Since \( T_C V_{11}^+ \subset C_{24} \cap (R_{24} \cup R_{20}) \), we define \( V_{13}^+ = \overline{V_{13}^+} \) as \( T_C V_{12}^- \), where \( T_C = P_2^+ \). Thus, \( \Phi = id \) on \( V_{10}^+ \) and \( V_{13}^+ \). The sets \( V_{11}^+ \) and \( V_{11}^- \) have been defined so that \( T_C V_{11}^+ = V_{12}^+ \) where \( T_C = A \), \( T_R V_{10}^+ = \overline{V_{11}^+} \) where \( T_R = P_1^+ AP_2^+ AP_1^+ \) and \( T_R \overline{V_{11}^-} = V_{12}^- \), where \( T_R = P_2^- \). Moreover, \( T_R V_{12}^- = \overline{V_{13}^-} \), where \( T_R = A \). Then the map \( \Phi \) from \( V_{11}^+ \) to \( \overline{V_{11}^+} \) and from \( V_{12}^- \) to \( \overline{V_{12}^-} \) can be defined so that the diagram commutes. We easily obtain \( \Phi = P_1^+ AP_2^+ A \) on \( V_{11}^+ \) and \( \Phi = AP_2^+ \) on \( V_{12}^- \). Note that in this case we only need \( A^2 = Id \).

The map \( \Phi \) on the pair of the sets \((V_{11}^+, V_{12}^-), (V_{21}^+, V_{22}^-) \) and \((V_{21}^-, V_{22}^+) \) can be defined in the same way by considering commutative diagrams. The map we obtain can be described as follows:

\[
\Phi = \begin{cases} 
AP_k^+ & \text{on } V_{k1}^+ \\
 AP_k^+ & \text{on } V_{k1}^-
\end{cases}
\]

where the matrices are

\[
P_1^+ AP_2^+ A = \begin{pmatrix} 1 & \pm(-1+i) \\ \pm(1-i) & -1 \end{pmatrix}, \quad P_2^+ AP_2^+ A = \begin{pmatrix} 1 & \pm(-1+i) \\ \pm(1+i) & -1 \end{pmatrix}.
\]

Note that \((P_1^+ AP_2^+ A)^2 = (P_2^+ AP_2^+ A)^2 = Id \) which we have used to determine \( \Phi = AP_k^+ \) on \( U_{11}^+ \).

We have proved the main result:

**Theorem 4.1.** — The rectilinear map is conjugate to the curvilinear map, that is, there exists a conjugacy map \( \Phi : C_1 \cup C_2 \to R_1 \cup R_2 \) satisfying \( \Phi \circ T_C = T_R \circ \Phi \) and \( \Phi = id \) on \((C_1 \cap R_1) \cup (C_2 \cap R_2)\).
We give the representation of the conjugacy map $\Phi$ by the coordinates $(\eta, \xi)_1$ and $(\eta, \xi)_2$:

$$
\Phi(\eta, \xi)_1 = \begin{cases}
\begin{pmatrix}
-1 & -1 \\
\eta - \sqrt{2} & \xi - \sqrt{2}
\end{pmatrix} & \text{on } U_{11}^+, V_{22}^-, \\
\begin{pmatrix}
-1 \\
\eta + \sqrt{2} & \xi + \sqrt{2}
\end{pmatrix} & \text{on } U_{11}^-, V_{22}^+, \\
\begin{pmatrix}
\eta - \sqrt{2} & \xi - \sqrt{2} \\
\sqrt{2} \eta - 1 & \sqrt{2} \xi - 1
\end{pmatrix} & \text{on } V_{11}^+, \\
\begin{pmatrix}
\eta + \sqrt{2} & \xi + \sqrt{2} \\
\sqrt{2} \eta + 1 & \sqrt{2} \xi + 1
\end{pmatrix} & \text{on } V_{11}^-
\end{cases}
$$

$$
\Phi(\eta, \xi)_2 = \begin{cases}
\begin{pmatrix}
-1 & -1 \\
\eta - \sqrt{2} & \xi - \sqrt{2}
\end{pmatrix} & \text{on } U_{21}^+, V_{12}^-, \\
\begin{pmatrix}
-1 & 0 \\
\eta + \sqrt{2} & \xi + \sqrt{2}
\end{pmatrix} & \text{on } U_{21}^-, V_{12}^+, \\
\begin{pmatrix}
\eta - \sqrt{2} & \xi - \sqrt{2} \\
\sqrt{2} \eta - 1 & \sqrt{2} \xi - 1
\end{pmatrix} & \text{on } V_{21}^+, \\
\begin{pmatrix}
\eta + \sqrt{2} & \xi + \sqrt{2} \\
\sqrt{2} \eta + 1 & \sqrt{2} \xi + 1
\end{pmatrix} & \text{on } V_{21}^-
\end{cases}
$$

**Remark.** We can also construct similar maps $T_C$, $T_R$ and $\Phi$ for the geodesic flow in the fundamental domain $F_4$ of the Hecke group $G_4$. We only write down results by using the action of matrices:

$$
T_C(\eta, \xi) = \begin{cases}
P^{-1}(\eta, \xi) & \text{on } C_1, \\
A(\eta, \xi) & \text{on } C_2 \cup C_3, \\
P(\eta, \xi) & \text{on } C_4,
\end{cases}
T_R(\eta, \xi) = \begin{cases}
P^{-1}(\eta, \xi) & \text{on } R_1, \\
A(\eta, \xi) & \text{on } R_2 \cup R_3, \\
P(\eta, \xi) & \text{on } R_4, \\
APA(\eta, \xi) & \text{on } R_5, \\
AP^{-1}A(\eta, \xi) & \text{on } R_6,
\end{cases}
$$

$$
\Phi(\eta, \xi) = \begin{cases}
AP(\eta, \xi) & \text{on } U_1^-, \\
AP^{-1}(\eta, \xi) & \text{on } U_1^+, \\
A^{-1}P(\eta, \xi) & \text{on } V_2^+, \\
A^{-1}P^{-1}(\eta, \xi) & \text{on } V_2^-, \\
APA(\eta, \xi) & \text{on } V_1^-, \\
AP^{-1}A^{-1}(\eta, \xi) & \text{on } V_1^+.
\end{cases}
$$

The notations of the domains of the maps are the same as before, except that we drop the number of the plane $k$, because in this case we use only one extended plane and the action of $A$ cannot switch the plane. If we do not use matrices, the representations of these maps are the same as the ones dropping $k$ from $T_C(\eta, \xi)_k$, $T_R(\eta, \xi)_k$ and $\Phi(\eta, \xi)_k$. 

References


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